

Kan Seminar

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You should read the masters! This seminar is about classical papers in algebraic topology.¹ The main goal of this seminar is to learn how to read (old) papers and to familiarize yourself with the fundamental results in algebraic topology that have been highly influential in the development of the field.

Some keywords that we will encounter are infinite symmetric products, Bott periodicity, transfer maps, the dual Steenrod algebra, Whitehead products, Massey products and Bousfield localizations. We conclude the seminar with four talks on algebraic K -theory in topology, ultimately ending with Quillen's computation of the K -theory of finite fields.

Each talk should be at most 80 minutes long, accounting for questions and comments. There is a lot of material in the references and you will not have enough time to discuss everything, so you should keep your talk survey-style. Try to find a suitable storyline for your talk and avoid technicalities. A well-chosen example is usually more instructive than a technical proof. You are left with some freedom to choose the content you want to include. We also included some more modern resources that might help you during your preparation, but certainly there is much more in the literature that might speak to you.

You should meet one of us at least 2 weeks prior to the talk and it can be useful to email us concrete math questions beforehand. This seminar should go well with the lecture course Algebraic Topology 1.

Syllabus. Here is a summary of the content and references.

1. **Quasifaserungen und unendliche symmetrische Produkte (19 October 2026).** Let $(X, 0)$ be a based space and set $\mathrm{SP}^n(X) := X^{\times n} / \Sigma_n$ for $n \geq 0$. There is an embedding

$$\mathrm{SP}^n(X) \hookrightarrow \mathrm{SP}^{n+1}(X), [x_1, \dots, x_n] \mapsto [x_1, \dots, x_n, 0].$$

Taking an infinite union yields the so-called *infinite symmetric product*

$$\mathrm{SP}(X, 0) := \bigcup_{n \geq 0} \mathrm{SP}^n(X) \quad \text{with the colimit topology.}$$

Dold and Thom proved $\pi_*(\mathrm{SP}(X, 0)) \cong \widetilde{H}_*(X; \mathbb{Z})$ for connected CW complexes X in [DT58]. Following Dold–Thom [DT58, Section 3, 4], you should define infinite symmetric products and the variant $\mathrm{AG}(X, 0)$. Mention Dold–Thom's theorem, it will be proven in the next talk. The proof relies on the notion of *quasifibrations*. You should introduce this and discuss basic properties and results [DT58, Section 1, 2].

Hatcher's treatment [Hat02, Section 4.K] might be helpful. Note that [DT58] is written in German. You should coordinate with the speaker of talk 2.

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¹Former MIT mathematician Daniel Kan was known to organize such a seminar over many years, whence the name.

2. **Quasifaserungen und unendliche symmetrische Produkte II (26 October 2026).** Building on talk 1, you should prove the Dold–Thom theorem, namely $\pi_*(\mathrm{SP}(X, 0)) \cong H_*(X; \mathbb{Z})$ for connected CW complexes X , see [DT58, Section 5, 6]. Deduce a theorem from Moore, namely that path-connected commutative and associative H -spaces with a strict identity element are weakly homotopy equivalent to a product of Eilenberg–MacLane spaces [DT58, Section 7].

Hatcher’s treatment [Hat02, Section 4.K] might be helpful. Note that [DT58] is written in German. You should coordinate with the speaker of talk 1.

3. **A new proof of the Bott periodicity theorem (2 November 2026).** Bott originally proved equivalences

$$\mathrm{BU} \times \mathbb{Z} \simeq \Omega^2(\mathrm{BU} \times \mathbb{Z}) \quad \text{and} \quad \mathrm{BO} \times \mathbb{Z} \simeq \Omega^8(\mathrm{BO} \times \mathbb{Z}).$$

using Morse theory methods [Bot59]. Atiyah in his obituary for Bott [Ati13, p. 6] said: “*This paper was a bombshell. The results were beautiful, far-reaching and totally unexpected.*” These equivalences are nowadays coined *Bott periodicity*. You should give an alternative proof of complex Bott periodicity following Behrens [Beh02]. It uses the notion of quasi-fibrations developed in the previous talks. Discuss the real version and sketch how the argument needs to be adapted.

4. **The transfer map and fiber bundles (9 November 2026).** If $f: M \rightarrow N$ is a map of compact oriented d -dimensional manifolds, then there is a wrong-way map on cohomology obtained using Poincaré duality:

$$f_!: H^n(M; \mathbb{Z}) \cong H_{d-n}(M; \mathbb{Z}) \xrightarrow{f_*} H_{d-n}(N; \mathbb{Z}) \cong H^n(N; \mathbb{Z}).$$

Becker and Gottlieb [BG75] generalize this to a fiber bundle $p: E \rightarrow B$ whose fiber F is a compact smooth manifold by constructing (after suspending often enough) a map $\tau: B_+ \rightarrow E_+$, where $(-)_+ = (-) \sqcup *$ adjoins a free basepoint. This is nowadays called the *Becker–Gottlieb transfer map*. It induces a map $\hat{\tau}: H^*(E) \rightarrow H^*(B)$. Construct this map and discuss that $\hat{\tau} \circ p^*$ is multiplication by the Euler characteristic $\chi(F)$ and the effect of p^* on cohomology after inverting the $\chi(F)$, see [BG75, Theorem 5.7].²

5. **The Steenrod algebra and its dual (16 November 2026).** Singular cohomology $H^*(-; \mathbb{F}_p)$ obtains a natural action from the so-called (p -primary) Steenrod algebra \mathcal{A} . It is an infinitely generated noncommutative ring subject to the Adem relations. Following Milnor [Mil58] you should show that the dual of the Steenrod algebra \mathcal{A}_* is a graded-commutative Hopf algebra, a kind of algebraic object with a multiplication and a comultiplication, which interact nicely. Determine its structure explicitly.

Rognes’ notes [Rog23] might be helpful.

6. **On the homotopy groups of the union of spheres (23 November 2026).** Whitehead products were originally defined in [Whi41], you should introduce them, e.g. following [Hat02, Example 4.52]. Follow Hilton to describe how the homotopy groups of a finite wedge of spheres splits into a certain infinite sum of specified homotopy groups of spheres using Whitehead products [Hil55, Theorem A]. This is the Hilton–Milnor theorem with Milnor deserving a mention since he proved a generalization [MAS72]. Mention the Jacobi identity of Whitehead products [Hil55, Theorem B] and discuss some applications [Hil55, Section 5].

²Spectra are briefly used in this article. If you are unfamiliar with them, you should imagine singular (co)homology, which is a special case.

Whitehead's discussion in [Whi78, Chapter XI] might be helpful. Moreover, there is a note on Whitehead products by Hilton and Whitehead [HW53] which might also be helpful. If you are comfortable with the language of ∞ -categories, you might find Lavenir's thesis [Lav25] enjoyable.

7. **Higher order linking numbers (30 November 2026).** The Borromean rings are three circles in space which cannot be separated from each other but which break apart into two unlinked circles once one of them is removed. This can be made mathematically rigorous using the so-called *Massey products* [Mas69]. Those were originally defined in [UM57, Mas58]. Follow [McC01, Section 8.2, Massey's triple product] to introduce Massey products and discuss how they can be used to treat the Borromean rings [Mas69].
8. **The localization of spaces with respect to homology (7 December 2026).** For a generalized homology theory E_* and a space X , the homology $E_*(X)$ only sees the part of X relevant to E . For example, $H_*(-; \mathbb{Q})$ does not detect torsion phenomena, so it cannot distinguish between $\mathbb{R}P^\infty$ and $*$.

You should follow Bousfield [Bou75] and set up for every generalized homology theory E_* a functor L_E on spaces equipped with a natural transformation $X \rightarrow L_E X$ witnessing $L_E X$ as the closest space to X containing information only seen by E . This is the *E-Bousfield localization*. You should select various examples from [Bou75] and discuss those.

9. **Finiteness Conditions for CW-complexes (14 December 2026).** For computations it is usually helpful to choose efficient CW cell structures on spaces, so finite ones are often handier than infinite ones. A *finitely dominated space* is a space X such that there exists a finite CW complex Y together with maps $i: X \rightarrow Y$ and $r: Y \rightarrow X$ with $r \circ i \simeq \text{id}_X$.

One may ask whether every (path-connected and locally simply-connected) finitely dominated space is homotopy equivalent to a finite CW complex. Wall characterized this through an obstruction $o(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1 X])$ in the zeroth reduced algebraic K -theory group [Wal65]. Introduce K_0 and discuss Wall's theorem.

A helpful reference for K_0 could be [Lan25, Section 3]. You should coordinate with the speaker of talk 10 about K_0 .

10. **On the cohomology and K -theory of general linear groups over a finite field I (11 January 2027).** In 1957 Grothendieck announced the zeroth algebraic K -theory group $K_0(R)$ of a ring R at the mathematische Arbeitstagungen in Bonn [Hir07]. While higher K -theory groups K_1 and K_2 were defined after, it seemed hard to extend this definition suitably until Quillen suggested a definition of all higher K -groups with his plus construction. He computed the algebraic K -theory of finite fields in [Qui72]: For $n \geq 1$ and q a prime power he showed

$$K_0(\mathbb{F}_q) \cong \mathbb{Z} \quad \text{and} \quad K_n(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1, \\ 0 & n \text{ even.} \end{cases}$$

As of today this remains one of the only full computations of algebraic K -theory groups. We will see this spectacular computation in the last three talks of the seminar.

You should mention that our goal for the last three talks is to compute $K_*(\mathbb{F}_q)$. The point of this talk is to familiarize the audience with the basics of algebraic K -theory. Follow [Lan25, Section 3] to introduce the lower algebraic K -theory groups K_0, K_1 and K_2 . Define the higher algebraic K -theory groups using Quillen's plus construction [Qui72, Section 12].

Kuku's notes [Kuk03] and Weibel's K -book [Wei13] might be useful. You should coordinate with the speaker of talk 9 about K_0 .

11. **On the cohomology and K -theory of the general linear groups over a finite field II (18 January 2027).** Let $q \in \mathbb{Z}$. Introduce the Adams operations $\Psi^q: BU \rightarrow BU$ and the space

$$F\Psi^q := \text{hofib}(1 - \Psi^q: BU \rightarrow BU)$$

from [Qui72, Section 1]. Describe its cohomology and its homology [Qui72, Theorem 1, Theorem 2].

Hebestreit–Wagner's lecture notes [HW21, Theorem III.19], Land's lecture notes [Lan25] and Weibel's K -book [Wei13] might be useful. There are also many talk notes available online about this topic. You should coordinate with the speaker of talk 12.

12. **On the cohomology and K -theory of general linear groups over a finite field III (25 January 2027).** Describe the homology and cohomology of $GL_n(\mathbb{F}_q)$ following Quillen [Qui72]. Conclude by computing the K -theory of finite fields [Qui72, Theorem 8].

Hebestreit–Wagner's lecture notes [HW21, Theorem III.19], Land's lecture notes [Lan25] and Weibel's K -book [Wei13] might be useful. There are also many talk notes available online about this topic. You should coordinate with the speaker of talk 11.

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